PC 13108-N

L-4/2111
FIELD THEORY—MM602/AMC—309
Semester—III
(Common for AMC/Math)

Time Allowed : Three Hours] [Maximum Marks : 70

Note :– The candidates are required to attempt *two* questions each from Section A and B. Section C will be compulsory.

SECTION—A

1. Let $F \subseteq E \subseteq K$ be fields. If $[K : E] < \infty$ and $[E : F] < \infty$, then prove that :

   (i) $[K : F] < \infty$ and

   (ii) $[K : F] = [K : E]\,[E : F]$.

2. Let E and F be fields and let $\sigma : F \to E$ be an embedding of F into E. Then prove that there exist a field K such that F is a subfield of K and σ can be extended to an isomorphism of K onto E.

3. (i) Define the splitting field of a polynomial by giving an example. Prove that upto isomorphism there is only one splitting field of a polynomial.

   (ii) Find the splitting field of $x^4 - 2 \in Q[x]$ over Q and its degree of extension.

4. Define a separable element and separable extension. Prove that if E is a finite separable extension of a field F, then E is a simple extension of F. $2\times10=20$

SECTION—B

5. Prove that the group $G(Q(\alpha)/Q)$ where $\alpha^5 = 1$ and $\alpha \neq 1$ is isomorphic to the cyclic group of order 4.

6. State and prove fundamental theorem of Galois theory.

7. Let F contain a primitive n^{th} root of unity. Then prove that the following statements are equivalents :

   (i) E is a finite cyclic extension of degree n over F.

   (ii) E is the splitting field of an irreducible polynomial $x^n - b \in F[x]$.

8. Prove that $f(x) \in F[x]$ is solvable by radicals over F if and only if its splitting field E over F has solvable Galois group G(E/F). $2\times10=20$

SECTION—C

9. Write in brief :

   (a) Show that $x^3 + 3x + 2 \in Z/(7)\,[x]$ is irreducible over the field Z/(7).

   (b) Let [E : F] = 2, where E is an extension of F then show that E is normal extension of F.

   (c) State Einstein's criterion for irreducible polynomial.

   (d) Show that $Q(\sqrt{2}, \sqrt{5}) = Q(\sqrt{2}+\sqrt{5})$.

   (e) State Kronecker's theorem.

   (f) Prove that Galois group of $x^4 + x^2 + 1$ is the same as that of $x^6 - 1$ and is of order 2.

   (g) Define cyclic extension and extension by radicals of field.

   (h) Express the symmetric function $x_1^2 + x_2^2 + x_3^2$ as rational functions of elementary symmetric functions.

   (i) State fundamental theorem of algebra.

   (j) Show that $x^3 + ax^2 + bx + 1 \in Z[x]$ is reducible over Z if and only if either a = b or a + b = –2. $10\times3=30$