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GROUP THEORY

Paper–302

Semester–III

(Common for MC & B.Sc.
Hons. in Mathematics) Part–II

Time Allowed : 3 Hours] [Maximum Marks : 70

Note : The candidates are required to attempt **two** questions each from Sections A and B carrying 10 marks each and the entire Section C consisting of 10 short answer type questions carrying 3 marks each.

SECTION—A

1. (a) Let G be a group and H and K be normal subgroups of G. Then prove that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G.

   (b) Let H be a non-empty finite subset of a group G. If H is closed under the operation of G, then prove that H is a subgroup of G.

2. State and prove the Fundamental theorem of Cyclic groups.

3. (a) Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show that $|a| \cap |b| = \{e\}$.

   (b) Find a noncyclic subgroup of A_8 that has order 4.

4. (a) Prove that the set of all 2 × 2 matrices with entries from $\mathbb{R}$ and determinant 1 is a group under matrix multiplication.

(b) Prove that in a group G, $(ab)^2 = a^2b^2$ if and only if ab = ba for all $a, b \in G$.

SECTION—B

5. If a group G is the internal direct project of a finite number subgroups $H_1, H_2, \ldots H_n$ then G is isomorphic to the external direct product of $H_1, H_2, \ldots H_n$.

6. Prove that for any prime p, every group of order p^2 is either isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

7. (a) State and prove Fermat's Little theorem.

   (b) State and prove Cauchy's theorem for abelian groups.

8. (a) Prove that if for a group G, $o(G) = p^k$, where p is prime and k is some integer, then $o(Z(G)) > 1$.

   (b) Let G be a group and Z(G) be its centre. If $\frac{G}{Z(G)}$ is cyclic, then show that G is abelian.

SECTION—C

9. Attempt all the following questions :

   (i) For a group G, define an isomorphism map from $\frac{G}{Z(G)}$ to Inn(G). Prove its homomorphism property.

   (ii) Prove that the alternative group A_5 does not have a subgroup of order 30.

   (iii) Show that G is abelian if and only if Z(G) = G.

   (iv) Let H be a subgroup of group G. If $x^2 \in H \ \forall \ x \in G$, then prove that H is a normal subgroup of G.

   (v) Define Quotient group. Prove that Quotient group of an abelian group is abelian.

   (vi) Show that Conjugacy relation is an equivanlence relation.

   (vii) What is the maximum order of any element in A_{10} ?

(viii) Find all of the left cosets of $\{1, 11\}$ in $U(30)$.

(ix) State the Fundamental theorem of finite abelian groups.

(x) How many distinct abelian groups (upto isomorphism) of order 100, exists.