PC-11761/NJ

D-6/2111

ADVANCED CALCULUS-231
Semester-III

Time : Three Hours] [Maximum Marks : 45

Note : Attempt *two* questions each from Section-A and B. Section-C will be compulsory.

SECTION – A

I. Prove that the countable union of countable sets is countable.

II. (a) Find the supremum and infimum of the set

$$S = \left\{ \pi + \frac{1}{2},\ \pi + \frac{1}{4},\ \pi + \frac{1}{8},\ \ldots\ldots\ldots \right\}.$$

(b) Show that the sequence

$$F_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots\ldots\ldots\ldots + \frac{1}{2n-1}$$

is not a Cauchy sequence. Is it convergent.

III. Prove that a bounded sequence $< F_n >$ converges to l if and only if

$$\underset{n\to\infty}{\text{lt}} \sup F_n = \underset{n\to\infty}{\text{lt}} \inf F_n = l.$$

IV. Examine the convergence of the series :

$$\sum_{n=1}^{\infty} \frac{2.4.6.....2n}{1.3.5......(2n+1)} .$$ (2×6=12)

SECTION – B

V. Show that $\sum (-1)^n \frac{n+2}{2^n+5}$ cos nx is convergent for all real values of x.

VI. Let f be a function defined on (0, 1) by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1/q, & \text{if } x = p/q \end{cases}$$

where p and q are positive integers having no common factor. Prove that f is continuous at each irrational point and discontinuous at each rational point.

VII. Find the maximum value of the function

$$f(x) = x^2 e^{-x}, x > 0 \cdot$$

VIII. (a) Test if Lagrange's mean value theorem holds for the function $f(x) = |x|$ in the interval [–1, 1].

(b) Show that $f(x) = x \tan^{-1}(y_x)$ for $x \neq 0$ and $f(0) = 0$ is not differentiable at $x = 0$. (2×6=12)

SECTION – C

IX. (a) Find $\underset{n\to\infty}{\text{lt}} \dfrac{\sin n\pi}{n}$.

(b) Let $a_1 = 1$, $a_{n+1} = \sqrt{7a_n}$, $n \geq 1$. Find $\lim_{n\to\infty} a_n$.

(c) Show that the series $\sum_{n=1}^{\infty} \cos(1/n^2)$ is not convergent.

(d) Let $a_k, b_k \in \mathrm{R}$ such that $|a_k| \leq b_k \; \forall \; k \in \mathbb{N}$ if $\sum b_k$ is convergent, then show that $\sum_k a_k$ is absolutely convergent.

(e) Show that $f(x) = x^2$ is uniformly continuous in [0, 1].

(f) Check for the differentiability of the function $f(x) = |x+2|$ at $x = -2$.

(g) Derive the expansion of cos x in terms of power series.

(7×3=21)