PC-13483/NJ

E-27/2111

CALCULUS OF SEVERAL VARIABLES AND IMPROPER INTEGRALS-502

(Semester-V)

Time : Three Hours] [Maximum Marks : 70

Note : Attempt any *two* questions each from Section A and B. Section C is compulsory.

SECTION – A

I. (a) Compute $f_{xy}(0, 0)$ for the function

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x+y^2} ; & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$ 5

(b) Prove that the existence of total derivative of a differentiable function at a point c guarantees existence of directional derivative for every u in $\mathbb{R}^n$. Is the converse true? Justify. 5

II. (a) Prove that for a real valued function f: $\mathbb{R}^n \to \mathbb{R}$ $f'(c)(v) = \nabla f(c).v$ for $v = v_1u_1 + v_2u_2 + ... + v_nu_n$, where u_1, u_2 ,.... u_n are unit coordinate vectors in $\mathbb{R}^{n}$. 5

(b) If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$

show that $\left(\frac{\partial\omega}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\omega}{\partial\theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$. 5

III. (a) Show that the function $f(x, y) = \begin{cases} \frac{x^2+y^2}{|x|+|y|} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$

is continuous at (0, 0) but its partial derivatives do not exist at (0, 0). 5

(b) Expand $f(x, y) = \tan^{-1} y/x$ in the neighbourhood of (1, 1). 5

IV. (a) Find the maximum value of $\left|\sum_{k=1}^{n} a_k x_k\right|$, if $\sum_{k=1}^{n} x_k^2 = 1$, using Lagrange's Method. 6

(b) Write the conditions for a real valued function f with continuous second order partial derivatives to have relative maxima, relative minima and saddle point at a point c in $\mathbb{R}^2$. 4

SECTION – B

V. State and prove Lebesgue's criterion for existence of a multiple Riemann integral. 10

VI. Show that $\iint x^{m-1} y^{n-1} dxdy$, over the positive quadrant of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is $\frac{a^m b^n}{2n} \beta\left(\frac{m}{2}, \frac{n}{2} + 1\right)$. 10

VII. (a) Find the necessary and sufficient condition for the convergence of integral $\int_0^{\infty} \frac{dx}{x^n}$. 5

(b) Test the convergence of $\int_0^{\infty} \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$. 5

VIII. (a) Show that $\int_0^{\infty} \frac{\sin x}{x} dx$ is convergent, but not absolutely. 6

(b) Evaluate $\int_0^{\infty} 2^{-9x^2} dx$ using Gamma function. 4

SECTION – C

IX. (a) Find local maxima, local minima and saddle point, if any, of the function

$f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4.$

(b) Find the directional derivative of the function

$f = x^2 - y^2 + 2z^2$ at P(1, 2, 3) in the direction of the line PQ, where Q is the point (5, 0, 4).

(c) Show that $f(x, y) = |x| + |y|$ is not differentiable at (0, 0).

(d) State sufficient condition for the equality of mixed partial derivatives.

(e) Find the normal vector to the surface $Z = \sqrt{x^2 + y^2}$ at the point (3, 4, 5).

(f) If f is continuous on $[a, b; c, d]$, then prove that

$$\int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx .$$

(g) Show that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

(h) State Dirichlet's and Abel's test for convergence of Improper integrals of first kind.

(i) Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^3 \sin\theta \cos\theta \, d\theta \, dr$.

(j) Prove that $\int_{-1}^{\infty} \frac{x+1}{(x+2)^2} dx = \frac{1}{20}$. (10×3=30)