Roll No. .... **Total No. of Pages : 3**

PC 11454-NH

BS/2111

ANALYSIS—I

Semester—III

Time Allowed : Three Hours] [Maximum Marks : 40

Note :– The candidates are required to attempt *two* questions each from Sections A and B. Section C will be compulsory.

SECTION—A

I. State and prove Cauchy's first theorem on limits. Does the converse hold ? Justify. 6

II. (a) If $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n} - \log n$, then prove that $\{a_n\}$ is convergent sequence. 3

(b) Using concept of sequential continuity, show that the function

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at x = 0. 3

III. (a) Show that the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$ is convergent for $-1 < x \leq 1$. 3

(b) Show that for the series :

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{3^2}+\frac{1}{5^3}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\ldots\ldots,$$ Cauchy's root Test indicates convergence but D'Alembert's ratio test is inconclusive. 3

IV. (a) Apply Weirstrass's M-Test to show that the series

$\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$ is uniformly convergent $\forall x \in R$. 3

(b) For the power series $\sum_{n=0}^{\infty} a_n x^n$, if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = r$, r may be finite or infinite, then the radius of convergence R is given by $R = \frac{1}{r}$. 3

SECTION—B

V. Prove that the lower Riemann-integral cannot exceed the upper Riemann-integral i.e. $\underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx$. 6

VI. Prove that every continuous function defined on a closed interval is Riemann-integrable. 6

VII. (a) Find the lower and upper Riemann sums for the function

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

on [–1, 1] by dividing it into n equal sub-intervals. 3

(b) Show that $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ is divergent. 3

VIII. Prove that the improper integral $\int_0^{\infty} x^{n-1}e^{-x}\,dx$ is convergent if and only if $n > 0$. 6

SECTION—C

IX. (a) State Cauchy's general principle of convergence for sequences.

(b) If $\sum_{n=1}^{\infty} a_n$ is convergent series, then prove that $\lim_{n\to\infty} a_n = 0$.

(c) Prove that sequence $\{a_n\}$ where $a_n = \frac{1}{n}$ is Cauchy sequence.

(d) Define Subsequence.

(e) Define norm of a partition of closed interval.

(f) Define Riemann integrable function.

(g) State Abel's test for convergence of improper integrals.

(h) Prove that improper integral $\int_1^{\infty}\frac{\sin x}{x}\,dx$ is convergent.

8×2=16