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7274/N

J–15/2110

DIFFERENTIAL GEOMETRY

Paper–MM 404/A

Semester–I

Time Allowed : 3 Hours] [Maximum Marks : 70

Note : Attempt **two** questions each from Sections A and B carrying 10 marks each and the entire Section C consisting of 10 short answer type questions carrying 3 marks each.

SECTION—A

1. Define the six surface patches for the unit sphere S^2 in therms of cartesian coordinates and hence explain how they give S^2 the structure of a surface. 10

2. (a) Define principal curvatures of a surface. State and prove Euler's theorem.

(b) For the second fundamental form :

$$L du^2 + 2M\, du dv + N\, dv^2$$

of a surface patch $\sigma(u, v)$ of a surface S, show that if p is a point in the image of σ and v, w $\in T_p(S)$, then

$< W(v), w > = Ldu(v)du(w) + M[du(v)dv(w) + du(w)dv(v)] + Ndv(v)dv(w)$, where W stands for the Weingarten Map. 5+5

3. (a) Show that the first fundamental form of a surface at a point defines an inner product on the tangent space of the surface at that point.

(b) Define the Gaussian curvature in terms of the principal curvatures. Show that the principal curvatures are tl e roots or t ie equation :

$$\begin{vmatrix} L-KE & M-KF \\ M-KF & N-KG \end{vmatrix} = 0$$

and the principal vectors corresponding to these curvatures are the tangent vectors $t = \xi\sigma_u + \eta\sigma_v$ such that :

$$\begin{bmatrix} L-KE & M-KF \\ M-KF & N-KG \end{bmatrix} = 0 \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 5+5

4. (a) Prove that :

$$\begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{bmatrix} = J^t \begin{bmatrix} E & F \\ F & G \end{bmatrix} J,$$

where J is the Jacobian matrix of ϕ.

(b) Discuss the concept of tangent plane. Prove that if $\sigma(u, v)$ is a surface patch, the set of linear combinations of σ_u and σ_v is unchanged when σ is reparametrised. 5+5

SECTION—B

5. (a) Let $\gamma(t)$ be a unit speed curve on the helicoid $\sigma(u, v) = (u \cos v, u \sin v, v)$. Show that $\dot{u}^2 + (1+u^2)\dot{v}^2 = 1$. Also show that if γ is a geodesic on σ, then $\dot{v} = \frac{a}{1+u^2}$.

(b) State and prove Gauss Remarkable theorem. 5+5

6. (a) Show that a compact surface with Gaussian curvature > 0 everywhere and constant mean curvature is a sphere.

(b) State and prove the Geodesic equations. 5+5

7. (a) Find the surface patch of the forms :

$du^2 + dv^2$ and $-du^2$.

(b) State and prove the Codazzi-Mainardi equations. 5+5

8. (a) Let $\sigma : U \to R^3$ be a minimal surface patch and assume that $A_\sigma(u) < \infty$. Let $\lambda \neq 0$ and assume that the principal curvatures κ of σ satisfy $|\lambda\kappa| < 1$ everywhere, so that the parallel surface σ^λ of σ is a regular surface patch. Prove that $A_{\sigma\lambda}(u) \leq A_\sigma(u)$ and equality holds for some $\lambda \neq 0$ iff $\sigma(U)$ is an open subset of a plane.

(b) Prove that if a surface patch has fundamental form : $e^\lambda(du^2 + dv^2)$, where λ is a smooth function of u and v, then its Gaussian curvature K satisfies $\Delta\lambda + 2Ke^\lambda = 0$, where Δ denotes the Laplacian operator $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$. 5+5

SECTION—C

9. (a) Every open subset of a smooth surface is smooth.

(b) Find the arc length reparametrization of the curve $\gamma(t) = (a \cos t, a \sin t, bt)$.

(c) Prove that if $\tilde{\gamma}$ is reparametrization of a curve γ, then γ is a reparametrization of $\tilde{\gamma}$.

(d) Show that every helicoid is a minimal surface.

(e) Discuss the geometrical approach to the tangent vector.

(f) Show that the Codazzi-Mainardi equation reduce to :

$$L_v = \frac{1}{2}E_v\left(\frac{L}{E}+\frac{N}{G}\right) \text{ and } N_v = \frac{1}{2}G_u\left(\frac{L}{E}+\frac{N}{G}\right)$$

if the first fundamental form is $Edu^2 + Gdv^2$ and second fundamental form is

$Ldu^2 + N\,dv^2$.

(g) Prove that a smooth map $f : S_1 \to S_2$ is a local isometry if and only if the symmetric bilinear forms <.> and f^* <.> on $T_p S_1$ are equal for all $P \varepsilon S_1$.

(h) Discuss the effect of dilation on the second fundamental form of a surface.

(i) Calculate the principal curvatures of a unit sphere $\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$ and a helicoid $\sigma(u, v) = (v \cos u, v \sin u. \lambda_u)$

(j) Prove that for a diffeomorphism $f : S_1 \to S_2$, if σ_1 is an allowable surface patch on S_1, then $f \circ \sigma_1$ is an allowable surface patch on S_2.

10×3=30