10847/N

M-73/2110

DIFFERENTIAL GEOMETRY -MM 404/AMC 105

(Sem-I)

Time allowed: 3 hours Max. Marks: 70

Note: Attempt five questions in all selecting two questions each from Section A and B carrying 10 marks each and the compulsory Section C consisting of ten short answer type questions having 3 marks each.

SECTION-A

Q.I.(a) Discuss the geometrical interpretation of curvature of a plane curve.

(b) For the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ of a surface patch $\sigma(u, v)$ of a surface S, prove that if p is a point in the image of σ and $\mathbf{v}, \mathbf{w} \in T_p(S)$, then

$$< \mathbf{v}, \mathbf{w} >= Edu(\mathbf{v})du(\mathbf{w}) + F[du(\mathbf{v})dv(\mathbf{w}) + du(\mathbf{w})dv(\mathbf{v})] + Gdv(\mathbf{v})dv(\mathbf{w})$$

(5+5)

Q.II. Define the six surface patches for the unit sphere S^2 in therms of cartesian coordinates and hence explain how they give S^2 the structure of a surface. (10)

Q.III.(a)Let $\tilde{\sigma}(\tilde{u}, \tilde{v})$ be a reparametrisation of a surface patch $\sigma(u, v)$ with reparametrisation map $(u, v) \to \phi(\tilde{u}, \tilde{v})$. Show that the second fundamental form is left unchanged by a reparametrisation of the patch which preserves its orientation. Also prove that $\begin{bmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{bmatrix} = \pm J^t \begin{bmatrix} L & M \\ M & N \end{bmatrix} J$ where J is the Jacobian matrix of ϕ.

(b) Define a surface. Prove that every open subset of a surface is a surface. (5+5)

Q.IV.(a) Discuss the Frenet-Serret formulae for space curves.

(b) If γ is a unit speed curve on an oriented surface S, then prove that its Normal curvature $K_n =< W(\dot{\gamma}), \dot{\gamma} >$, where W stands for the Weingarten map of S at p. If σ is a surface patch of S and $\gamma(t) = \sigma(u(t), v(t))$ is a curve in σ, then show that $K_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$. (5+5)

SECTION-B

Q.V.(a) Prove that if a surface patch has fundamental form: $e^{\lambda}(du^2 + dv^2)$, where λ is a smooth function of u and v, then its Gaussian curvature K satisfies $\Delta\lambda + 2Ke^{\lambda} = 0$, where Δ denotes the Laplacian operator $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

(b) Define a geodesic. Prove that any normal section of a surface is a geodesic (5+5)

Q.VI.(a) Prove that on the surface of revolution $\sigma(u, v) = (f(u)cosv, f(u)sinv, g(u))$,

(i) Every meridian is a geodesic,

(ii) A parallel $u = u_\circ$ is a geodesic iff $\frac{df}{du} = 0$ when $u = u_\circ$ i.e. $u_\circ$ is stationary point of f.

(b) Prove that any point of a surface of constant Gaussian curvature is contained in a patch that is isometric to an open subset of a plane, a sphere or a pseudosphere. (5+5)

Q.VII. Let $\sigma : U \to R^3$ be a minimal surface patch and assume that $\mathcal{A}_\sigma(\mathcal{U}) < \infty$. Let $\lambda \neq 0$ and assume that the principal curvatures κ of σ satisfy $| \lambda\kappa |< 1$ everywhere, so that the parallel surface σ^λ of σ is a regular surface patch. Prove that $\mathcal{A}_{\sigma^\lambda}(\mathcal{U}) \leq \mathcal{A}_\sigma(\mathcal{U})$ and equality holds for some $\lambda \neq 0$ iff $\sigma(U)$ is an open subset of a plane. (10)

Contd-

Q.VIII. (a) Using Codazzi- Mainardi equations, show that there is no syface patch whose first fundamental from and second fundamental forms are $du^2 + cos^2u\,dv^2$ and $cos^2u\,du^2 + dv^2$.

(b) Prove that if a surface patch has fundamental form: $e^{\lambda}(du^2 + dv^2)$, where λ is a smooth function of u and v. Then its Gaussian curvature K satisfies $\Delta\lambda + 2Ke^{\lambda} = 0$, where Δ denotes the laplacian operator $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ **(5+5)**

SECTION-C

Q.IX.(a) If the two smooth surfaces S and $\bar{S}$ are diffeomorphic and S is orientable, then $\bar{S}$ is also orientable.

(b) Prove that if σ is a surface patch of an oriented surface S, then the matrix of the Weingarten map W with respect to the basis $\{\sigma_u, \sigma_v\}$ of $T_p(S)$ is $F_I^{-1}F_{II}$ where $F_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ and $F_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$

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(c) Prove that any local isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

(d) Show that the curve $\gamma(t) = \sigma(u(t), v(t))$ on a surface σ of S is a line of curvature iff $(EM - FL)\dot{u}^2 + (EN - GL)\dot{u}\dot{v} + (FN - GM)\dot{v}^2 = 0$

(e) Find the unit speed reparametrization of the curve $\gamma(t) = (cos^2t, sin^2t)$.

(f) If the tangent vector of a parametrized curve is constant, the image of the curve is a (part of)straight line.

(g) If K_1 and K_2 are the principal curavtures of a surface , the show that the Gaussain curvature $= K_1K_2$ and the the Mean curvature= $\frac{1}{2}(K_1 + K_2)$.

(h) Discuss the effect of dilation of R^3 on the first fundamental form of a surface S.

(i) Find the second fundamental form of a unit cylinder $\sigma(u, v) = (cosv, sinv, u)$.

(j) Prove that every open subset of a smooth surface is smooth. **(10×3=30)**

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