M-73/2110
Subject: Mathematics

10846/N

Title of the Paper: Topology-I (MM403/AMC-103)

Time Allowed: 3 hours

Maximum Marks: 70

Note:

1. Attempt **FIVE** questions in all by selecting **TWO** questions from each of the **Section A and B**.
2. **Section C is compulsory.**

SECTION - A

1. (a) Prove that for any cardinal number, there exists a larger cardinal number. (5)

   (b) Prove that for any set X, card 2^X= card $\mathcal{P}(X)$. (5)

2. (a) Prove that a set A is open if and only if $A = intA$. (5)

   (b) Prove that $\mathcal{B}$ is a basis for a topology τ on X if and only if for each $G \in \tau$ and each $x \in G$ there is a $U \in \mathcal{B}$ such that $x \in U \subset G$. (5)

3. (a) Let X be a set and $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ be a map such that

   i. $\eta(X) = X$,

   ii. $\eta(A) \subset A$,

   iii. $\eta \circ \eta(A) = \eta(A)$,

   iv. $\eta(A \cap B) = \eta(A) \cap \eta(B)$.

   Then $\tau = \{\eta(A) : A \in \mathcal{P}(X)\}$ is a topology and $int(A) = \eta(A)$. (5)

Contd—

(b) Let Y be a subspace of X. If $A \subset Y$ is open in Y, and Y is open in X, then A is open in X. (5)

4. (a) Prove that a bijective map $f : X \to Y$ is homeomorphism if and only if $f(\overline{A}) = \overline{f(A)}$ for each $A \subset X$. (5)

(b) Let $f : X \to Y$ be an open map. Given any subset S of Y and any closed set U containing $f^{-1}(S)$, there exists a closed set V containing S such that $f^{-1}(V) \subset U$. (5)

SECTION - B

5. (a) Let $\{Y_\alpha : \alpha \in \Lambda\}$ be any family of spaces and $f : X \to \prod_\alpha Y_\alpha$ be a map. Then f is continuous if and only if $p_\beta \circ f$ is continuous for each $\beta \in \Lambda$. (5)

(b) The cartesian product topology in $\prod_\alpha Y_\alpha$ is the smallest topology for which all projections $p_\beta : \prod_\alpha Y_\alpha \to Y_\beta$ are continuous. (5)

6. (a) Each continuous real valued function on a connected space takes on all values between any two that it assumes. (5)

(b) Prove that X is connected if and only if no continuous map $f : X \to 2$ is surjective. (5)

7. (a) If X is an ordered set in which every closed interval is compact, then X has the least upper bound property. (5)

(b) Let X be locally compact. Then its one-point compactification $\hat{X}$ is metrizable if and only if X is 2^0 countable. (5)

8. (a) Show that every metrizable space with a countable dense subset has a countable basis. (5)

(b) Let $P : X \to Y$ be a closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$. Show that if X is Hausdorff, then so is Y. (5)

SECTION - C

9. (a) Describe the open sets if all straight lines in the plane parallel to the x-axis are used for sub-basis.

   (b) Prove that $Fr(A) = \emptyset$ if and only if A is both open and closed set.

   (c) If every countable subset of X is closed, is the topology necessarily discrete?

   (d) Show that the set of rational numbers is not locally compact.

   (e) For what spaces X is the only dense set X itself?

   (f) Prove that the cardinal number of set of all continuous real valued functions on $\mathcal{R}$ is c.

   (g) Each path component of a space X is open if and only if each point of X has a path connected neighbourhood.

   (h) Prove that $\prod A_\alpha$ is dense in $\prod Y_\alpha$ if and only if $A_\alpha \subset Y_\alpha$ is dense.

   (i) Prove that the product of two Lindelof spaces need not be Lindelof.

   (j) Prove that the continuous image of a compact space is compact.

(10*3=30)

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