PC-10054/NJ

F-42/2110

ALGEBRA GROUP AND RING THEORY – 501
Semester-V

Time : Three Hours] [Maximum Marks : 70

Note : Attempt *two* questions each from Section A and B. Section C will be compulsory. Each question in Section A and B will be of 10 marks and section C will be of 30 marks.

SECTION – A

I. (a) Let G be a group having a composition series and H, a normal subgroup of G. Prove that G has a composition series, one of whose terms is H.

(b) Let H be a normal subgroup of a group G. If both H and G/H are solvable then G is solvable.

II. State and prove Zassenhaus Lemma.

III. (a) Show that any finite non-zero integral domain is a field.

(b) Show that intersection of two subrings is a subring.

IV. (a) Show that for two ideals A and B of a ring R, $A \bigcup B$ is an ideal of R if and only if either $A \subseteq B$ of $B \subseteq A$.

(b) Prove that the sum of two nilpotent ideals is nilpotent however the sum of two nilpotent elements may not be nilpotent.

SECTION – B

V. (a) Let $f : R \rightarrow R_1$ be a ring homomorphism then show that

(i) $f(0) = 0$ and

(ii) for $\forall\ 0 \in R$, $f(-a) = -f(a)$

(b) State and prove Fundamental theorem of Homomorphism.

VI. Show that every integral domain can be imbedded in a field.

VII. For any two non-zero polynomials $f = (a_{0,}\ a_1,\ a_2,\ .....)$, $g = (b_0,\ b_1,\ b_2,\ .....)$ over a ring R then prove that

(i) if $f + g \neq 0$ then $\deg (f + g) \leq \max (\deg f, \deg g)$.

(ii) if $fg \neq 0$ then $\deg (fg) \leq \deg f + \deg g$.

(iii) if R is a ring without proper zero-divisor, then $\deg fg = \deg f + \deg g$.

VIII. (a) Show that every Euclidean Domain is a Principal Ideal Domain.

(b) Show that in a unique Factorization Domain every irreducible element is prime.

SECTION – C

(Compulsory Question)

IX. (a) Define Subnormal and Normal series.

(b) Define Solvable group with example.

(c) Show that division ring is a simple ring.

(d) Define left ideal and right ideal of a ring R.

(e) Define Prime ideal and Maximal ideal.

(f) If f is a homomorphism of R into R′ ; them show that $f(0) = O'$ where O is the zero element of R and O′ is the zero element of R′.

(g) Define unique factorisation domain.

(h) If R is commutative then R[x] is commutative.

(i) Show that the following ring is Euclidean domain

$$Z\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} / a,\ b\ \in Z\right\}.$$

(j) If R is a ring with identity 1 and f is a homorphism of R into a ring R′ then show that $f(1)$ is the identity of R′.