PC-1682/M

L-4/2050

THEORY OF LINEAR OPERATORS–MM 702/AMC-411

(Semester–IV)

(Common for Math/AMC)

Time : Two Hours] [Maximum Marks : 70

Note : Attempt any *four* questions. All questions carry equal marks.

I. Two matrices representing the same linear operator T on a finite dimensional normed space X relative to any two bases for X are similar.

II. State and prove spectral mapping theorem for polynomials.

III. Let A be a complex Banach algebra with identity e. Then for any $x \in$ A, the spectrum $\sigma(x)$ is compact, and the spectral radius satisfies $r_\sigma(x) \leq \| x \|$.

IV. Let X and Y be normed spaces and T: X $\rightarrow$ Y be a linear operator. Then T is compact if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Tx_n\}$ in Y which has a convergent subsequence.

V. Let $T: X \to Y$ be a linear operator. If T is compact, then its adjoint operator $T^{*}:: Y' \to X'$ is also compact. Here X and Y are normed spaces and X' and Y' the dual spaces of X and Y.

VI. Let $T : X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then equations $T_\lambda x = 0$ and $T_\lambda{}^{*} f = 0$ have the same number of linearly independent solutions.

VII. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then:

(a) All the eigenvalues of T (if they exist) are real.

(b) Eigenvectors corresponding to (numerically) different eigenvalues of T are orthogonal.

VIII. If two bounded self adjoint linear operators S and T on a Hilbert space H are positive and commute ($ST = TS$), then their product ST is positive.

IX. (a) Define regular value and resolvent set.

(b) Define positive operator and positive square root of an operator.

(c) Define spectral family.

(d) Define projection. Show that a bounded linear operator $P : H \to H$ on a Hilbert space H is a projection if P is self-adjoint and idempotent.

(e) Define Fredholm alternative.

(f) Define Biorthogonal systems.

(g) Let $T : X \to X$ be a compact linear operator and $S : X \to X$ a bounded linear operator on a normed space X. Then TS is compact.

(h) If $x \in A$ is invertible and commutes with $y \in A$, show that x^{-1} and y also commute.

(i) Define Banach algebra.

(j) Define spectral radius.